APPENDIX A: Governing Equations and Low-Reynolds-Number Eddy Viscosity Model

The continuity equation:

$$\frac{\partial (\rho U)}{\partial x} + \frac{\partial (\rho V)}{\partial y} = 0$$

The momentum equations:

$$\frac{\partial (\rho UU)}{\partial x} + \frac{\partial (\rho VU)}{\partial y} = -\frac{dP}{dx} + \frac{\partial}{\partial x}\left[2(\mu + \mu_t)\frac{\partial U}{\partial x}\right] + \frac{\partial}{\partial y}\left[(\mu + \mu_t)\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right]$$

$$\frac{\partial (\rho UV)}{\partial x} + \frac{\partial (\rho VV)}{\partial y} = -\frac{dP}{dy} + \frac{\partial}{\partial x}\left[(\mu + \mu_t)\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[2(\mu + \mu_t)\frac{\partial V}{\partial y}\right] + B_T$$

where $B_T = g\beta(T - T_{ref})$ is the buoyancy term.

The k-equation:

$$\frac{\partial (\rho Uk)}{\partial x} + \frac{\partial (\rho Vk)}{\partial y} = \frac{\partial}{\partial x}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x}\right] + \frac{\partial}{\partial y}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + P_k + P_g - \rho\varepsilon + D$$

The ε-equation:

$$\frac{\partial (\rho U\varepsilon)}{\partial x} + \frac{\partial (\rho V\varepsilon)}{\partial y} = \frac{\partial}{\partial x}\left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x}\right] + \frac{\partial}{\partial y}\left[\left(\mu + \frac{\mu_t}{\sigma_\epsilon}\right)\frac{\partial \varepsilon}{\partial y}\right] +$$
$$c_{\epsilon 1} f_1 \frac{\varepsilon}{k}(P_k + P_g) - c_{\epsilon 2} f_2 \rho\frac{\varepsilon^2}{k} + E$$

The energy equation:

$$\frac{\partial (\rho UT)}{\partial x} + \frac{\partial (\rho VT)}{\partial y} = \frac{\partial}{\partial x}\left[\left(\frac{\mu}{\mathrm{Pr}} + \frac{\mu_t}{\sigma_t}\right)\frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[\left(\frac{\mu}{\mathrm{Pr}} + \frac{\mu_t}{\sigma_t}\right)\frac{\partial T}{\partial y}\right]$$

The turbulent kinetic energy production due to the mean strain is

$$P_k = \mu_t\left[2\left(\frac{\partial U}{\partial x}\right)^2 + 2\left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)^2\right]$$

and

$$P_g = \overline{\rho' v} g = -\rho\beta\overline{v\theta} g = \beta g \frac{\mu_t}{\sigma_t}\frac{\partial T}{\partial y}$$

is the expression for the turbulent kinetic energy production due to buoyancy, and

$$\mu_t = \rho c_\mu f_\mu \frac{k^2}{\varepsilon}$$

is the expression for the turbulent dynamic viscosity.

Constants and functions of the standard high-Reynolds-number $k-\varepsilon$ model are:

$$\begin{array}{llllll} c_\mu = 0.09; & c_{\varepsilon 1} = 1.44; & c_{\varepsilon 2} = 1.92; & \sigma_k = 1.0; & \sigma_e = 1.3; & \sigma_t = 0.9 \\ f_\mu = 1.0; & f_1 = 1.0; & f_2 = 1.0; & D = 0; & E = 0 & \end{array}$$

Low-Reynolds-Number Model

The above $k-\varepsilon$ model can be applied to free flows, but cannot be used explicitly for near-wall flows. In order to apply it in the vicinity of the wall either a low-Reynolds-number or a wall-function approach needs to be used.

The various parameters in the LRN Launder-Sharma $k-\varepsilon$ model take the following form:

$$f_\mu = \exp\left(\frac{-3.4}{(1+\mathrm{Re}_t/50)^2}\right); \qquad f_1 = 1.0; \qquad f_2 = 1 - 0.3\exp(-\mathrm{Re}_t^2), \text{ where } \mathrm{Re}_t = \frac{\rho k^2}{\mu\varepsilon}$$

$$D = -2\mu\left[\left(\frac{\partial\sqrt{k}}{\partial x}\right)^2 + \left(\frac{\partial\sqrt{k}}{\partial y}\right)^2\right];$$

$$E = 2\frac{\mu\mu_t}{\rho}\left[\left(\frac{\partial^2 U}{\partial x^2}\right)^2 + \left(\frac{\partial^2 U}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 U}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2 V}{\partial x^2}\right)^2 + \left(\frac{\partial^2 V}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 V}{\partial x\partial y}\right)^2\right]$$

The wall boundary conditions for k and ε are: $k_w = 0$, $\varepsilon_w = 0$.

The D term in the k-equation allows us to set the value of ε at the wall to zero, whereas in reality the wall dissipation rate is

$$\varepsilon_w = \mu\left[\overline{\left(\frac{\partial u}{\partial r}\right)^2} + \overline{\left(\frac{\partial w}{\partial r}\right)^2}\right]$$

That is to say, the D term is equal to the dissipation rate in the immediate vicinity of the wall and its influence is negligible in regions where the turbulent Reynolds number is high. The E term is an additional dissipation source term, which makes a contribution only across the 'buffer layer'.

The remaining constants in the model have been assigned their values after many computational tests on free turbulent flows. However, in the vicinity of the wall c_μ is not a constant due to the decay of the turbulent shear stress. Similarly, $c_{\varepsilon 2}$ changes its value towards the wall to reflect the decay of turbulence at low Reynolds numbers. Thus, in the 'Low-Reynolds-Number' model these constants are functions of the turbulent Reynolds number Re_t. Functions f_μ and f_2 reflect these dependencies.